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The braid group representations associated with some non-fundamental representations of Lie algebras

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Abstract. The braid group representations associated with the eight-dimensional representation of B_3 and six-dimensional representation of A_2, including the quantum Lie algebra connected with the latter, are explicitly calculated.

1. Introduction

Recently much progress has been made in deriving the braid group representations (BGR). To find BGR the different types of approaches have been formulated [1-16]. One of them is based on reducing the known quantum $R(x)$ matrix (where x relates to the spectral parameter u). For instance, Wadati *et al* [9] derived the BGR and link polynomials other than Jones' by taking the limit of the known $R(x)$ matrices for spin models (under the symmetry breaking transformation). Turaev discussed the cases of B_n, C_n and D_n in their fundamental representations by reducing Jimbo's results on $R(x)$ matrices associated with the generalised Toda lattice [10]. Another type of approach is direct calculation. They are typically state models, especially the Yang-Baxter state model [6, 7, 15], and the general constructions of BGR for simple Lie algebras by Reshetikhin [11]. We know that the derivations of $R(x)$ are much more difficult than those of BGR which is denoted by S ($=R(0)$ in Jimbo's 'gauge' [17-19]). Meanwhile from the point of view of physics we prefer the direct calculation to the reduction versions.

In general it needs a lot of computation to find BGR as the dimensions of the Lie algebras are large. For instance, in the Yang-Baxter state model the state expansions are very complicated [7, 9, 15] and in the group approach [11] the calculations of the projectors are also lengthy. In comparison with Jimbo's works [17-19] the direct derivation approaches are almost on the same level as the 'full' quantum group (QG) version [13]. Therefore the key point for constructing BGR is to facilitate the computations in the direct derivation approach.

In this paper we present a more practical approach for the direct derivation of BGR in which the Witten and Wilson loop theory of link polynomials based on a (2+1) Chern-Simons Lagrangian plays the central role. We shall combine the decompositions of direct products for Lie algebra representations, Yang-Baxter state expansions [7, 11, 15], and the Markov trace [9] with Witten's version [14, 20, 21] to calculate two BGR corresponding to the eight-dimensional representations of B_3 and six-dimensional

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representation of SU(3). These two examples are beyond the discussions restricted to only the fundamental representations of Lie algebras as made in the current literature.

2. The universality of Witten's approach

The original discussions on the link theory based on the (2+1)-dimensional Chern–Simons Lagrangian are concerned with the fundamental representations of SU(N). However, by making comparisons with other projection theories it is easy to see that the approach works for any decomposition of $R\otimes R=\oplus_i E_i$ which give rise to the eigenvalues of BGR [14, 20, 21]

$$\lambda_i=\pm\bar{q}^{(\Delta_R-\frac{1}{2}\Delta_{E_i})} \tag{2.1}$$

where

$$\bar{q}=\exp[2\mathrm{i}\pi/(k+C_v)]$$

and Δ stands for the corresponding Casimir eigenvalues.

The reduction relation for the BGR S is

$$\prod_{i=1}^{n}(S-\lambda_i)=0. \tag{2.2}$$

The framing factor f is given by

$$\langle\chi S^m\psi\rangle=f^m P(L_{m-1}) \tag{2.3}$$

where $P(L_{m-1})$ denotes $(m-1)$ crossing polynomials. The meaning of $\langle\chi\ldots\psi\rangle$ and other conventions are referred to in [14, 21]. The skein relation has the form

$$P(L_{n-1})-f^{-1}\left(\sum_{i=1}^{n}\lambda_i\right)P(L_{n-2})+f^{-2}\left(\prod_{i<j}^{n}\lambda_i\lambda_j\right)P(L_{n-3})$$
$$-\ldots+f^{-n}(-1)^n\left(\prod_{i=1}^{n}\lambda_i\right)P(L_{-1})=0. \tag{2.4}$$

The Markov trace is defined as

$$\Phi(A)=\mathrm{Tr}(A\bar{H}) \tag{2.5}$$

where the matrix A can be any crossing block and $\bar{H}$ is given by

$$\bar{H}=\bar{h}\times\bar{h}\times\ldots\times\bar{h}\qquad \bar{h}=h/[\bigcirc].$$

The diagonal matrix h is given by [9, 11]

$$h_{ab}=\delta_{ab}t^{-2\langle\delta,W_a\rangle}=\delta_{ab}t^{-L(a)} \tag{2.6}$$

where δ is the half sum of simple roots and W_a denotes the weight labelled by index a. We note that the sum

$$\sum_a S^{ab}_{ab}h_{bb}=\tau\qquad \sum_a (S^{-1})^{ab}_{ab}h_{bb}=\bar{\tau} \tag{2.7}$$

is independent of index a. The normalisation of S^{aa}_{aa} can be made in an arbitrary way. For example

$$S^{aa}_{aa}=1 \tag{2.8}$$

or

$$S^{aa}_{aa} = t^{-(W_R)^2} \tag{2.9}$$

which gives rise to

$$\sum_b S^{ab}_{ab} h_{bb} = t^{-\Delta_R} = \tau \tag{2.10}$$

$$\sum_b (S^{-1})^{ab}_{ab} h_{bb} = t^{\Delta_R} = \bar{\tau} = \tau^{-1}. \tag{2.11}$$

W_R denotes the highest weight of representation R. The factor

$$\alpha^{1/2} = (\bar{\tau}/\tau)^{1/2} = t^{\Delta_R}$$

plays an important role in the construction of polynomials just as f in (2.3). Obviously, in comparison with Witten's approach we have

$$f = \alpha^{-1/2} \tag{2.12}$$

if this condition holds:

$$t = \bar{q} \tag{2.13}$$

which is known as the trace-cross-channel unitarity [7, 15].

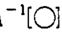
2.1. *Kauffman's state model*

The link polynomial is defined by

$$P_K = (A)^{-W(K)} \sum_S [K|\tilde{S}] t^{\|\tilde{S}\|} \tag{2.14}$$

where

[] $= A$ []

[] $= A^{-1}$[]. (2.15)

The notations adopted here refer to [7, 15]. For example the norm $\|\tilde{S}\|$ is given by

$$\|\tilde{S}\| = \sum_{\mathrm{Comp}(S)} \mathrm{rot}(a)\langle 2\delta W_a\rangle \tag{2.16}$$

where a suitable label set $\{a\}$ is understood [12, 22].

Because the calculations show

$$A = \tau$$

and the writhe

$W($$) = -m$

$W($$) = m$ (2.17)

(2.14) is equivalent to the definition of link polynomials in [7].

There are two possibilities for closure of the graph

 .

One of them is trivial:

The other provides the trace-cross-channel unitarity

$$
\begin{aligned}
&= \sum_{a,b,c} S^{ca}_{ca}(S^{-1})^{cb}_{cb} t^{-L(a)} t^{-L(b)} t^{-L(c)} \\
&= \sum_{c} \left(\sum_{a} S^{ca}_{ca} t^{-L(a)} \right) \left[\sum_{b} (S^{-1})^{cb}_{cb} t^{L(b)} \right] t^{-L(c)} \\
&= \sum_{c} t^{-L(c)} = [\bigcirc]. \qquad (2.18)
\end{aligned}
$$

Thus the trace-cross-channel unitarity is automatically satisfied, provided $t = q$.

Now, throughout the above discussions we have pointed out that the three approaches are closely related and sometimes they are equivalent. The Witten approach is more powerful not only in the explicit forms of the eigenvalues but also in their connections with the CFT.

In the following we shall employ any one of the three approaches when it is able to simplify the calculations. As we see later this 'combining operation' is effective in deriving BGR and is a general method.

3. The explicit derivation of BGR associated with an eight-dimensional representation of B_3

In order to carry out the arguments of [11, 14] we first note that for the decomposition

$$
\begin{array}{cccccccc}
 & D^{(0,0,\frac{1}{4})} \otimes & D^{(0,0,\frac{1}{4})} = & D_s^{(0,0,\frac{1}{2})} \oplus & D_A^{(0,\frac{1}{2},0)} + & D_A^{(\frac{1}{2},0,0)} \oplus & D_S^{(0,0,0)} \\
\text{Dim} & \mathbf{8} & \mathbf{8} & \mathbf{35} & \mathbf{21} & \mathbf{7} & \mathbf{1}
\end{array} \qquad (3.1)
$$

there are the weight vectors

$$
\begin{array}{lll}
W_7 = (0, 0, \frac{1}{4}) & W_5 = (0, \frac{1}{2}, -\frac{1}{4}) & W_3 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}) \\
W_1 = (\frac{1}{2}, 0, -\frac{1}{4}) & W_{-1} = (-\frac{1}{2}, 0, \frac{1}{4}) & W_{-3} = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{4}) \\
W_{-5} = (0, -\frac{1}{2}, \frac{1}{4}) & W_{-7} = (0, 0, -\frac{1}{4}). &
\end{array} \qquad (3.2)
$$

They obey the weight conservations

$$W_a + W_b = W_c + W_d \qquad (3.3)$$

and are grouped by:

(a) 8-fold weights, 4-fold weights

$$
\begin{aligned}
&W_a + W_{-a} = 0 \\
&W_5 + W_3 = W_7 + W_1 = (\tfrac{1}{2}, 0, 0) \\
&W_5 + W_{-1} = W_7 + W_{-3} = (-\tfrac{1}{2}, \tfrac{1}{2}, 0) \\
&W_7 + W_{-5} = W_3 + W_{-1} = (0, -\tfrac{1}{2}, \tfrac{1}{2}) \\
&W_5 + W_{-7} = W_1 + W_{-3} = (0, \tfrac{1}{2}, -\tfrac{1}{2}) \\
&W_3 + W_{-7} = W_1 + W_{-5} = (\tfrac{1}{2}, -\tfrac{1}{2}, 0)
\end{aligned}
$$

(*b*) 2-fold weights

$$
\begin{array}{ll}
W_5+W_7=(0,\tfrac{1}{2},0) & W_3+W_{-5}=(\tfrac{1}{2},-1,\tfrac{1}{2}) \\
W_7+W_3=(\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}) & W_{-1}+W_{-3}=(-1,\tfrac{1}{2},0) \\
W_7+W_{-1}=(-\tfrac{1}{2},0,\tfrac{1}{2}) & W_{-5}+W_{-1}=(-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}) \\
W_5+W_1=(\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}) & W_1+W_{-7}=(\tfrac{1}{2},0,-\tfrac{1}{2}) \\
W_3+W_1=(1,-\tfrac{1}{2},0) & W_{-3}+W_{-7}=(-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}) \\
W_5+W_{-3}=(-\tfrac{1}{2},1,-\tfrac{1}{2}) & W_{-5}+W_{-7}=(0,-\tfrac{1}{2},0)
\end{array}
$$

(*c*) single-fold weights

$$W_a+W_a.$$

With the above weight analysis the BGR S has the block diagonal form

$$S=\text{block diag}(A_1,A_2,\ldots,A_7,A_8,A_7,\ldots,A_2,A_1)$$

where

$$
A_1=u \qquad A_2=\begin{bmatrix} 0 & p_1 \\ p_1 & w_1 \end{bmatrix} \qquad A_3=\begin{bmatrix} 0 & 0 & p_1 \\ 0 & u & 0 \\ p_1 & 0 & w_1 \end{bmatrix} \qquad A_4=\begin{bmatrix} & & & p_2 \\ & & p_3 & q_1 \\ & p_3 & w_3 & q_2 \\ p_2 & q_1 & q_2 & w_2 \end{bmatrix}
$$

$$
A_5=\begin{bmatrix} & & & & p_1 \\ & & & p_1 & \\ & & u & & \\ & p_1 & & w_1 & \\ p_1 & & & & w_1 \end{bmatrix} \qquad A_6=\begin{bmatrix} & & & & & p_4 \\ & & & & p_5 & q_3 \\ & & & p_6 & 0 & 0 \\ & & p_6 & w_6 & 0 & 0 \\ & p_5 & 0 & 0 & w_5 & q_4 \\ p_4 & q_3 & 0 & 0 & q_4 & w_4 \end{bmatrix}
$$

$$
A_7=\begin{bmatrix} & & & & & & p_7 \\ & & & & & p_8 & 0 \\ & & & & p_9 & 0 & q_5 \\ & & & u & 0 & 0 & 0 \\ & & p_9 & 0 & w_9 & 0 & q_6 \\ & p_8 & 0 & 0 & 0 & w_8 & 0 \\ p_7 & 0 & q_5 & 0 & q_6 & 0 & w_7 \end{bmatrix}
$$

$$
A_8=\begin{bmatrix} & & & & & & & p_{10} \\ & & & & & & p_{11} & q_7 \\ & & & & & p_{12} & q_{13} & q_8 \\ & & & & p_{13} & q_{17} & q_{14} & q_9 \\ & & & p_{13} & 0 & q_{18} & q_{15} & q_{10} \\ & & p_{12} & q_{17} & q_{18} & w_{12} & q_{16} & q_{11} \\ & p_{11} & q_{13} & q_{14} & q_{15} & q_{16} & w_{11} & q_{12} \\ p_{10} & q_7 & q_8 & q_9 & q_{10} & q_{11} & q_{12} & w_{10} \end{bmatrix}
$$

where the vacancies mean 0 in the submatrices.

The eigenvalues can be determined by the Casimirs

$$\lambda_1 = q^3 \qquad \lambda_2 = -q^{-1} \qquad \lambda_3 = -q^{-9} \qquad \lambda_4 = q^{-21}$$

and the framing factor is

$$f = t^{-21}$$

with

$$t = q$$

where

$$q = \exp\left(\frac{1}{4}\frac{\mathrm{i}\pi}{C_v + k}\right) = t.$$

We note that (2.13) is the consequence of the trace-cross-channel unitarity.

In terms of the Witten version we write the eigenvalue equation and the skein relation immediately:

$$(S - t^3)(S - t^{-1})(S - t^{-9})(S - t^{-21}) = 0$$

and

$$\begin{aligned} P(L_{3+}) = {} & t^{18}(1 - t^4 - t^{12} + t^{24})P(L_{2+}) + t^{40}(1 + t^8 + t^{24} + t^{32} - t^{12} - t^{20})P(L_+) \\ & - t^{112}P(L_-) + t^{70}(1 - t^{12} - t^{20} + t^{24})P(L_0). \end{aligned} \tag{3.4}$$

The Markov trace under this case has come from

$$h_{ab} = \delta_{ab} t^{L(a)} \qquad L(a) = \begin{cases} t^{4a+2} & a > \frac{3}{2} \\ t^{4a} & |a| \leqslant \frac{3}{2} \\ t^{4a-2} & a < -\frac{3}{2} \end{cases}$$

and

$$[\bigcirc] = \sum_a t^{L(a)} = t^{18} + t^{14} + t^6 + t^2 + t^{-2} + t^{-6} + t^{-14} + t^{-18}$$

with the label set

$$a, b \in (\tfrac{7}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}, -\tfrac{5}{2}, -\tfrac{7}{2}).$$

The general considerations preserve

$$\sum_b (S^{-1})^{ab}_{ab} h_{bb} = t^{-21} \qquad \sum_b S^{ab}_{ab} h_{bb} = t^{21} \tag{3.5}$$

namely

$$\tau = t^{-21} \qquad \bar{\tau} = t^{21}.$$

We next determine the elements in the submatrices.

(i) From the eigenvalues λ_i it is easy to know that

$$\begin{aligned} & w_1 = q^3 - q^{-1} && w_2 + w_3 = 2q^3 - q^{-1} - q^{-9} \\ & w_4 + w_5 + w_6 = 3q^3 - 2q^{-1} - q^{-9} && w_7 + w_8 + w_9 = 3q_3 - 2q^{-1} - q^{-9} \\ & w_{10} + w_{11} + w_{12} = 3q^3 - 3q^{-1} - q^{-9} + q^{-21}. \end{aligned}$$

(ii) Equation (3.5) leads to

$$a=\tfrac{7}{2} \qquad q^3q^{18}=A=q^{21}$$

$$a=\tfrac{5}{2} \qquad q^3q^{14}+w_1q^{18}=A$$

$$a=\tfrac{3}{2} \qquad q^3q^6+w_3q^{14}+w_1q^{18}=A$$

i.e. $w_3=w_1(1+q^{-4})$ and $w_2=w_1(1+q^{-8})$

$$a=\tfrac{1}{2} \qquad q^3q^2+w_6q^6+w_1q^{14}+w_2q^{18}=A \Rightarrow w_6=w_1.$$

(iii) Two diagrams without annihilations give

$$w_9=w_5=w_3 \qquad w_4=w_2=w_7 \qquad w_1=w_6=w_8.$$

(iv) Using (iii) and the Markov trace we have

$$a=-\tfrac{1}{2} \Rightarrow w_{13}=0$$

$$a=-\tfrac{3}{2} \Rightarrow w_{12}=w_1(1-q^{-8})$$

$$a=-\tfrac{5}{2} \Rightarrow w_{11}=w_1(1-q^{-16})$$

$$a=-\tfrac{7}{2} \Rightarrow w_{10}=w_1(1+q^{-8}-q^{-12}-q^{-20}).$$

(v) The other parameters, p and q, can be determined by the 37 diagrams with annihilations.

Noting that there are only six independent conditions provided by (3.5) we thus have $6+2+37=45$ constraint conditions plus one equality in total which matches the number of parameters appearing in the submatrices A_i $(2+13+13+18=46)$. Putting it all together we derive the final answers:

$$u=q \tag{3.6}$$

and

$$\begin{gathered}
p_1=p_6=p_8=u \qquad p_2=p_3=p_4=p_5=p_7=p_8=p_9=u^{-1} \\
p_{10}=p_{11}=p_{12}=p_{13}=u^{-3} \\
w_1=u^3-u^{-1} \qquad w_2=w_1(1+u^{-8}) \qquad w_3=w_1(1+u^{-4}) \qquad w_4=w_7=w_2 \\
w_5=w_3=w_9 \qquad w_6=w_8=w_1 \qquad w_{10}=w_2(1-u^{-12}) \qquad w_{11}=w_2(1-u^{-8}) \\
w_{12}=w_1(1-u^{-8}) \qquad w_{13}=0 \\
q_1=u^{-2}w_1 \qquad q_2=-u^{-4}q_1 \qquad q_3=q_5=q_1 \\
q_4=q_6=q_2 \qquad q_7=u^{-4}w_1 \qquad q_8=-u^{-8}w_1 \qquad q_9=u^{-2}w_2 \\
q_{10}=u^{-12}w_1 \qquad q_{11}=-u^{-6}w_2 \qquad q_{12}=u^{-10}w_2 \qquad q_{13}=u^{-2}w_3 \\
q_{14}=-u^{-8}w_1 \qquad q_{15}=-u^{-6}w_3 \qquad q_{16}=u^{-12}w_1 \qquad q_{17}=u^{-4}w_1 \\
q_{18}=u^{-2}w_3.
\end{gathered} \tag{3.7}$$

Thus, based on the Witten version and Reshetikhin discussions [11, 14] we know that the eight-dimensional case of B_3 is similar to the case of G_2 as discussed in [11].

4. The representation of the braid group and the polynomials associated with the six-dimensional representation of A_2

We now turn to carry out the arguments shown in [11, 14] for the case of six-dimensional representations of A_2. In this case the decomposition is

$$6 \otimes 6 = \oplus E_i = 15_s \oplus 15_a \oplus 6_s.$$

The corresponding eigenvalues are λ_1, λ_2 and λ_3, respectively. The corresponding Casimirs are

$$\Delta_R = \tfrac{20}{3} \qquad \Delta_1 = \tfrac{32}{3} \qquad \Delta_2 = \tfrac{56}{3} \qquad \Delta_3 = \tfrac{20}{30}$$

which give the eigenvalues

$$\lambda_1 = v^4 \qquad \lambda_2 = -v^{-2} \qquad \lambda_3 = v^{-5} \tag{4.1}$$

where

$$v = \exp\left(-\frac{2\pi i}{3}\frac{4}{C_v + k}\right). \tag{4.2}$$

The framing factor is given by

$$\begin{aligned} &\sum_b S^{ab}_{ab} h_{bb} = A = v^{10} \\ &\sum_b (S^{-1})^{ab}_{ab} h_{bb} = A^{-1} = v^{-10} \end{aligned} \tag{4.3}$$

where

$$h_{aa} = v^{L(a)}$$

and

$$L(a) = \begin{cases} \frac{3}{2}(a/2+1) & a = 6, 2 \\ 1 & a = 0, -2 \\ \frac{3}{2}a + 3 & a = -4, -6. \end{cases}$$

Using the eigenvalues and the framing factor it is easy to write out the skein relation in the form

$$P(L_{2+}) = \alpha^{1/2}\beta P(L_+) + \alpha\gamma P(L_0) + \alpha^{3/2}\delta P(L_-)$$

where

$$\beta = v^4 - v^{-2} + v^{-5} \qquad \gamma = v^2 - v^{-1} + v^{-7} \qquad \delta = -v^{-3}$$

$$\alpha = \bar{\tau}/\tau = v^{-20}.$$

Taking $v = \bar{t}^{-1}$ we obtain

$$P(L_{2+}) = \bar{t}^{10}(\bar{t}^5 - \bar{t}^2 + \bar{t}^{-4})P(L_+) + \bar{t}^{20}(\bar{t}^7 - \bar{t} + \bar{t}^{-2})P(L_0) - \bar{t}^{33}P(L_-). \tag{4.4}$$

To determine the explicit form of the BGR we first specify the weight conservation conditions. The weight vectors under the case are

$$W_6=\tfrac{2}{3}(2,-1,-1) \qquad W_2=\tfrac{1}{3}(1,1,-2)$$

$$W_0=\tfrac{1}{3}(1,-2,1) \qquad W_{-2}=\tfrac{2}{3}(-1,2,-1)$$

$$W_{-4}=\tfrac{1}{3}(-2,1,1) \qquad W_{-6}=-\tfrac{2}{3}(1,1,-2)$$

where the label set of the weights has been taken to be $(6, 2, 0, -2, -4, -6)$ in accordance with the weight conservation.

To write down the block diagonal submatrices of S we list the requirements due to the weight conservation as the following:

W_6+W_6	$1\otimes 1$ matrix		
W_6+W_2	$2\otimes 2$ matrix	W_6+W_0	$2\otimes 2$ matrix
$2W_2=W_6+W_{-2}$	$3\otimes 3$ matrix	$W_6+W_{-4}=W_2+W_0$	$4\otimes 4$ matrix
$2W_0=W_6+W_{-6}$	$5\otimes 5$ matrix	$W_2+W_{-6}=W_0+W_{-4}$	$5\otimes 5$ matrix
$W_2+W_{-4}=W_0+W_{-2}$	$4\otimes 4$ matrix		
$W_{-6}+W_0$	$(\neq)$	$W_{-4}+W_{-2}$	$4\otimes 4$ matrix
$W_{-6}+W_{-2}=2W_{-4}$			$3\otimes 3$ matrix
$W_{-6}+W_{-4}$			$2\otimes 2$ matrix
$W_{-6}+W_{-6}$			$1\otimes 1$ matrix

We therefore have the general form:

$$S=\text{block Diag}(A_1, A_2^{(1)}, A_2^{(2)}, A_3^{(1)}, A_4^{(1)}, A_5^{(1)}, A_4^{(2)}, A_5^{(2)}, A_4^{(3)}, A_3^{(2)}, A_2^{(3)}, A_1)$$

where

$$A_1=u_1 \qquad A_2^{(1)}=A_2^{(2)}=A_2^{(3)}=\begin{bmatrix} 0 & p_1 \\ p_1 & w_1 \end{bmatrix}$$

$$A_3^{(1)}=A_3^{(2)}=\begin{bmatrix} 0 & & p_2 \\ & u_2 & q_1 \\ p_2 & q_1 & w_2 \end{bmatrix} \qquad A_4^{(1)}=\begin{bmatrix} 0 & & & p_3 \\ & & p_4 & q_2 \\ & p_4 & w_4 & q_3 \\ p_3 & q_2 & q_3 & w_3 \end{bmatrix}$$

$$A_4^{(2)}=\begin{bmatrix} 0 & & & p_7 \\ & & p_8 & q_2 \\ & p_8 & 0 & q_2 \\ p_7 & q_2 & q_2 & w_7 \end{bmatrix} \qquad A_4^{(3)}=\begin{bmatrix} 0 & & & p_1 \\ & & p_1 & 0 \\ & p_1 & w_1 & 0 \\ p_1 & 0 & 0 & w_1 \end{bmatrix}$$

$$A_5^{(1)}=\begin{bmatrix} 0 & & & & p_5 \\ & & & p_6 & 0 \\ & & u_2 & 0 & q_5 \\ & p_6 & 0 & w_6 & 0 \\ p_5 & 0 & q_5 & 0 & w_5 \end{bmatrix} \qquad A_5^{(2)}=\begin{bmatrix} 0 & & & & p_3 \\ & & & p_4 & q_2 \\ & & u_1 & 0 & 0 \\ & p_4 & 0 & w_4 & q_3 \\ p_3 & q_2 & 0 & q_3 & w_3 \end{bmatrix}.$$

The same method can be used to determine the parameters appearing in the submatrices A_i as done in section 3.

First we take the traces of A_i that give

$$w_1=\lambda_1+\lambda_2 \qquad u_2+W_2=\lambda_1+\lambda_2+\lambda_3$$

$$w_3+w_4=\lambda_1+2\lambda_2+\lambda_3$$

$$u_2+w_5+w_6=2\lambda_1+2\lambda_2+\lambda_3 \qquad w_7=\lambda_1+2\lambda_2+\lambda_3.$$

Equation (4.3) proves six-constraint conditions and the parameters p and q should be determined by the diagrams with annihilations. Putting all the solutions together we have

$$\begin{aligned}
&u_1=v^4 \qquad u_2=v \qquad w_1=w_6=v^4-v^{-2} \qquad w_2=w_3=w_5=w_1(1-v^{-3})\\
&w_4=v-v^{-2} \qquad w_7=v^4-2v^{-2}+v^{-5}\\
&p_1=p_6=v \qquad p_2=p_3=p_5=p_8=v^{-2} \qquad p_4=p_7=v^{-1/2}\\
&q_1=v^{-3/2}w_1 \qquad q_2=v^{-3/2}w_4(1+v^3)^{1/2}\\
&q_3=v^{3/2}q_2 \qquad q_5=v^{-3/2}w_1=q_1.
\end{aligned} \tag{4.5}$$

It is able to compute the simple link polynomials such as

 $=\bar{t}^{18}(\bar{t}^{-12}+\bar{t}^{-3}+1+\bar{t}^{6}+\bar{t}^{9}+\bar{t}^{12})$

 $=\bar{t}^{45}-\bar{t}^{39}-\bar{t}^{36}-\bar{t}^{27}+\bar{t}^{24}+\bar{t}^{21}+\bar{t}^{12}.$

We thus complete the braid group representations associated with the six-dimensional representation of A_2 based on combining the general discussion with Witten's approach and the extended diagrammatic calculations [11, 14, 15].

5. The associated quantum algebras

The general theory was discussed first by Reshetikhin, Takhtajan and Faddeev, see [13, 22], based on the algebraic functions of quantum n-dimensional vector space C_q^n associated with the relation

$$f(\hat{R})(x\otimes x)=0 \tag{5.1}$$

where $f(\hat{R})$ is a polynomial of $\hat{R}=PR$ as the usual notation.

We now want to give the quantum Lie algebras associated with six-dimensional representation of SU(3) using the RTF method [22]. The $\hat{R}$ here is the braid group representation S in our notation. Supposing S has been known, for example, by the general derivation as shown in [22] or its concrete computation with the aid of Witten's approach [14, 20], the projectors are then given. For the given eigenvalues λ_1 (with multiplicity m_1), ..., λ_n (with multiplicity m_n) the diagonal matrix is denoted by

$$\Lambda=\mathrm{diag}(\underbrace{\lambda_1,\ldots,\lambda_1}_{m_1},\underbrace{\lambda_2,\ldots,\lambda_2}_{m_2},\ldots,\underbrace{\lambda_n,\ldots,\lambda_n}_{m_n})$$

with which the submatrices $A_{(a)}$ can be expressed by

$$A_{(a)} = L\Lambda^{(a)}L^{t} \tag{5.2}$$

$$A_{(a)(\alpha)(\delta)} = \sum_{i=1}^{n} \lambda_i \sum_{\beta=\Sigma_1^i m_{i-1}+1}^{\Sigma_1^i m_i} L^{(a)}_{(\alpha)(\beta)} L^{(a)}_{(\beta)(\delta)}$$

$$= \sum_{i=1}^{n} \lambda_i P^{(a)}_{i(\alpha)(\delta)} \tag{5.3}$$

where $(\alpha), (\beta), \ldots,$ are the double indices of the matrix S and (a) indicates the submatrix. Note that $m_0 = 0$.

The $P_i^{(a)}$ stand for the projectors relating with the submatrices $A_{(a)}$:

$$P_i^{(a)} = \prod_{j\neq i}^{n} (A_{(a)} - \lambda_j)\left(\prod_{j\neq i}^{n} (\lambda_i - \lambda_j)\right)^{-1}. \tag{5.4}$$

Following RTF it is easy to write the quantum Lie algebras with the help of L^t where L^t is the transpose of L. For instance, in the case of spin 1 (O(3)) the eigenvalues are $\lambda = t$, $-t$ and t^{-2} and correspondingly for the largest submatrix

$$L^t = \begin{bmatrix} 1/a^{1/2} & -t^{1/2}(1+t)/a^{1/2} & t^2/a^{1/2} \\ 1/b^{1/2} & t^{-1/2}(1-t)/b^{1/2} & -1/b^{1/2} \\ t/c^{1/2} & t^{1/2}/c^{1/2} & 1/c^{1/2} \end{bmatrix} \tag{5.5}$$

where

$$a = 1 + t(1+t)^2 + t^4 \qquad b = t + t^{-1} \qquad c = 1 + t + t^2.$$

Keeping in mind the implications of the label set of S it is easy to read the corresponding commutation relations of the quantum algebras. They are

$$\text{for } \lambda_1 = t \qquad x_1 x_{-1} + t^2 x_{-1} x_1 = t^{1/2}(1+t)x_0^2$$

$$\text{for } \lambda_2 = -t^{-1} \qquad x_1 x_{-1} - x_{-1} x_1 = -t^{-1/2}(1-t)x_0^2$$

$$\text{for } \lambda_3 = t^{-2} \qquad t x_1 x_{-1} + x_{-1} x_1 = -t^{1/2} x_0^2$$

where $t = q^{-1}$, in comparison to [22].

Now we turn to the case of section 4. For convenience in the following we use $\bar{S} = v^{-1}S$ instead of S and put

$$v^3 = t$$

in (4.5). Corresponding to the submatrices $\bar{A}_2, \bar{A}_3, \bar{A}_4^{(1)}, \bar{A}_4^{(2)}, \bar{A}_4^{(3)}, \bar{A}_5^{(1)}$, and $\bar{A}_5^{(2)}$ we have

$$\bar{L}_2 = \frac{1}{(1+t^2)^{1/2}} \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$$

$$\bar{L}_3 = \begin{bmatrix} 1/a' & -1/b' & t/c' \\ (1/a')t^{1/2}(1+t) & (1/b')t^{-1/2}(1-t) & -t^{1/2}/c' \\ t^2/a' & 1/b' & 1/c' \end{bmatrix}$$

where

$$a' = (1 + t + 2t^2 + t^3 + t^4)^{1/2}$$

$$b' = (t + t^{-1})^{1/2} \qquad c' = (1 + t + t^2)^{1/2}.$$

By specifying the labelled set $(6, 2, 0, -2, -4, -6)$ we obtain

$$\left.\begin{aligned} x_6x_0+tx_0x_6&=0\\ x_6x_2+tx_2x_6&=0\end{aligned}\right\}\qquad \text{for } \bar\lambda_1=t$$

$$\left.\begin{aligned} -tx_6x_2+x_2x_6&=0\\ -tx_6x_0+x_0x_6&=0\end{aligned}\right\}\qquad \text{for } \bar\lambda_2=-t^{-1}$$

and

$$x_6x_{-2}+t^{1/2}(1+t)x_2x_2+t^2x_{-2}x_6=0 \qquad \text{for } \bar\lambda_1=t$$

$$-x_6x_{-2}+t^{-1/2}(1-t)x_2x_2+x_{-2}x_6=0 \qquad \text{for } \bar\lambda_2=-t^{-1}$$

$$tx_6x_{-2}-t^{1/2}x_2x_2+x_{-2}x_6=0 \qquad \text{for } \bar\lambda_3=t^{-2}.$$

Because of the multiplicities of the eigenvalues appearing in the submatrices, such as the $\bar A_4$ type and $\bar A_5$ type, we have to perform the orthogonalisations to the corresponding eigenvalue vectors. The results are shown as the following:

$$\bar L_4^{(1)}=\begin{bmatrix} t^{-1}/\alpha & 0 & -(1+t)^{1/2}/\gamma & t^2/\delta \\ \frac{1}{\alpha}t^{-1/2}(1+t)^{1/2} & -\frac{t^{1/2}}{\beta} & \frac{1}{\gamma}t^{1/2}(t^{-1}-1) & -\frac{t^2}{\delta}(1+t)^{-1/2} \\ \frac{1}{\alpha}(1+t)^{1/2} & \frac{1}{\beta} & \frac{1-t}{\gamma} & -\frac{t^{3/2}}{\delta}(1+t)^{-1/2} \\ \frac{t}{\alpha} & 0 & \frac{1}{\gamma}(1+t)^{1/2} & \frac{t}{\delta}\end{bmatrix}$$

$$\bar L_4^{(2)}=\begin{bmatrix} \frac{t^{-1}}{\alpha}(1+t)^{1/2} & 0 & -\frac{t^{1/2}}{d}(1+t)^{1/2} & \frac{t^2}{e} \\ \frac{1}{\alpha} & -\frac{1}{\sqrt2} & \frac{t^{1/2}}{2d}(t^{-1}-t) & -\frac{t}{e}(1+t)^{1/2} \\ \frac{1}{\alpha} & \frac{1}{\sqrt2} & \frac{t^{1/2}}{2d}(t^{-1}-t) & -\frac{t}{e}(1+t)^{1/2} \\ \frac{t^{1/2}}{\alpha}(1+t)^{1/2} & 0 & \frac{1}{d}(1+t)^{1/2} & \frac{t^{1/2}}{e}\end{bmatrix}$$

where

$$\alpha^2=t^2+t+2+t^{-1}+t^{-2}$$

$$\beta^2=1+t \qquad \gamma^2=t^2+t+1+t^{-1} \qquad \delta^2=t^4+t^3+t^2$$

$$d^2=\tfrac12(t+t^{-1})(1+t)^2 \qquad e^2=t^4+2t^3+2t^2+t.$$

$$\bar L_4^{(3)}=\frac{1}{\rho}\begin{bmatrix} 1 & 0 & 0 & -t \\ 0 & 1 & -t & 0 \\ 0 & t & 1 & 0 \\ t & 0 & 0 & 1\end{bmatrix}$$

where $\rho^2=1+t^2$.

$$\bar{L}_5^{(1)} = \begin{bmatrix} \frac{1}{a} & 0 & -\frac{1}{b} & 0 & \frac{t}{c} \\ 0 & \frac{1}{\rho} & 0 & -\frac{t}{\rho} & 0 \\ \frac{t^{1/2}}{a}(1+t) & 0 & \frac{t^{-1/2}}{b}(1-t) & 0 & -\frac{t^{1/2}}{c} \\ 0 & \frac{t}{\rho} & 0 & \frac{1}{\rho} & 0 \\ \frac{t^2}{a} & 0 & \frac{1}{b} & 0 & \frac{1}{c} \end{bmatrix}$$

$$\bar{L}_5^{(2)} = \begin{bmatrix} \frac{t^{-1}}{\alpha} & 0 & 0 & -\frac{(1+t)^{1/2}}{\gamma} & \frac{t^2}{\delta} \\ \frac{t^{-1/2}}{\alpha}(1+t)^{1/2} & -\frac{t^{1/2}}{\beta} & 0 & \frac{t^{1/2}(t^{-1}-1)}{\gamma} & \frac{-t^2}{\delta(1+t)^{1/2}} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{(1+t)^{1/2}}{\alpha} & \frac{1}{\beta} & 0 & \frac{1-t}{\gamma} & \frac{-t^{3/2}}{\delta(1+t)^{1/2}} \\ \frac{t}{\alpha} & 0 & 0 & \frac{(1+t)^{1/2}}{\gamma} & \frac{t}{\delta} \end{bmatrix}.$$

For $\bar{\lambda}_1 = t$:

$$t^{-1}x_6x_{-4} + t^{-1/2}(1+t)^{1/2}x_2x_0 + (1+t)^{1/2}x_0x_2 + tx_{-4}x_6 = 0$$
$$t^{-1}(1+t)^{1/2}x_2x_{-4} + x_0x_{-2} + x_{-2}x_0 + t^{1/2}(1+t)^{1/2}x_{-4}x_2 = 0$$
$$x_0x_{-6} + tx_{-6}x_0 = 0 \qquad x_{-2}x_{-4} + x_{-4}x_{-2} = 0$$
$$x_6x_{-6} + t^{1/2}(1+t)x_0 + t^2x_{-6}x_6 = 0$$
$$x_2x_{-2} + tx_{-2}x_2 = 0$$
$$t^{-1}x_2x_{-6} + t^{-1/2}(1+t)^{1/2}x_0x_{-4} + (1+t)^{1/2}x_{-4}x_0 + tx_{-6}x_2 = 0$$
$$x_{-2}x_{-2} = 0.$$

For $\bar{\lambda}_2 = -t^{-1}$:

$$-t^{1/2}x_2x_0 + x_0x_2 = 0$$
$$-(1+t)^{1/2}x_6x_{-4} + t^{-1/2}(1-t)x_2x_0 + (1-t)x_0x_2 + (1+t)^{1/2}x_{-4}x_6 = 0$$
$$-x_0x_{-2} + x_{-2}x_0 = 0$$
$$-t^{1/2}(1+t)^{1/2}x_2x_{-4} + t^{1/2}(t^{-1}-t)(x_0x_{-2} + x_{-2}x_0) + (1+t)^{1/2}x_{-4}x_2 = 0$$
$$-tx_{-2}x_{-4} + x_{-4}x_{-2} = 0 \qquad -tx_0x_{-6} + x_{-6}x_2 = 0$$
$$-tx_2x_{-2} + x_{-2}x_2 = 0 \qquad -x_6x_{-6} + t^{-1/2}(1-t)x_0x_0 + x_{-6}x_6 = 0$$
$$-t^{1/2}x_0x_{-4} + x_{-4}x_0 = 0$$
$$-(1+t)^{1/2}x_2x_6 + t^{1/2}(t^{-1}-1)x_0x_{-4} + (1-t)x_{-4}x_0 + (1+t)^{1/2}x_{-6}x_2 = 0.$$

For $\bar{\lambda}_3 = t^{-2}$:

$$tx_6x_{-4} - t(1+t)^{-1/2}x_2x_0 - t^{1/2}(1+t)^{-1/2}x_0x_2 + x_{-4}x_6 = 0$$

$$t^2x_2x_{-4} - t(1+t)^{1/2}x_0x_{-2} - t(1+t)^{1/2}x_{-2}x_0 + t^{1/2}x_{-4}x_2 = 0$$

$$t^{1/2}x_6x_{-6} + t^{-1/2}x_{-6}x_6 = x_0x_0$$

$$tx_2x_{-6} - t(1+t)^{-1/2}x_0x_{-4} - t^{1/2}(1+t)^{-1/2}x_{-4}x_0 + x_{-6}x_2 = 0.$$

We have thus listed all of the commutation relations obeyed by the quantum algebras associated with the six-dimensional representation of SU(3) in terms of the explicit forms of the projectors.

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